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QUALITATIVE PROPERTIES OF THE BALLISTIC TRAJECTORY.*

By T. H. Gronwall.

1. **Introduction.** In the following, we shall consider the projectile as a particle, and refer it to a right-handed system of rectangular coordinates with the origin at the muzzle of the gun, the positive x-axis being horizontal and directed toward the target and the positive y-axis vertical and directed upward. The retardation R due to the resistance of the air is introduced in its most general form as a function of the position and velocity of the projectile and the time, so that $R = R(x, y, z, x', y', z', t)$, where $x = dx/dt$, etc. Denoting by $v = \sqrt{x'^2 + y'^2 + z'^2}$ the velocity of the projectile, the components of the retardation are Rx'/v, Ry'/v, Rz'/v, and g being the acceleration of gravity, the differential equations of motion are

$$x'' = -\frac{Rx'}{v}, \qquad y'' = -\frac{Ry'}{v} - g, \qquad z'' = -\frac{Rz'}{v},$$

with the initial conditions $x = y = z = 0$, $x' = x_0' > 0$, $y' = y_0' > 0$, $z' = 0$ for $t = 0$. The first and third equations give $(z'/x')' = 0$, so that $z' = \text{const. } x'$, and since $z' = z = 0$ for $t = 0$, it follows that $z = 0$ for any t, that is, the trajectory lies in the xy-plane.

Consequently the velocity is given by

$$v = \sqrt{x'^2 + y'^2},$$

and introducing the notation

$$E = E(x, y, x', y', t) = \frac{R}{v},$$

the differential equations of the trajectory become

$$x'' = -Ex', \qquad y'' = -Ey' - g, \tag{1}$$

with the initial conditions for $t = 0$

$$x = y = 0, \qquad x' = x_0' > 0, \qquad y' = y_0' > 0. \tag{2}$$

For purposes of comparison, let us consider briefly the trajectory in vacuum, where $E = 0$. Equations (1) and (2) then give

$$x' = x_0', \qquad y' = y_0' - gt, \qquad x = x_0't, \qquad y = y_0't - \tfrac{1}{2}gt^2. \tag{3}$$

Besides the origin, two points of the trajectory are of special interest:

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* Read before the American Mathematical Society, April 26, 1919 and February 28, 1920.

the summit, or point of maximum ordinate, and the point of fall, where $y = 0$. The coördinates, velocity components and time will be denoted by the subscript s at the summit and by the subscript ω at the point of fall.

By τ we denote the angle of slope defined by $\tan \tau = dy/dx$ and $-\pi/2 < \tau < \pi/2$, so that $x' = v \cos \tau$, $y' = v \sin \tau$; the value of τ for $t = 0$ is called the *angle of departure* and denoted by α, and for $t = t_\omega$, we write $\tau_\omega = -\omega$ and call ω the *angle of fall.* The abscissa x_ω of the point of fall is the *range* of the trajectory, and the corresponding t_ω is the *time of flight.* Eliminating t in (3), we obtain

$$y = \tan \alpha \cdot x - \frac{gx^2}{2x_0'^2}, \tag{4}$$

and since the summit and the point of fall are defined analytically by $y_s' = 0$ and $y_\omega = 0$ respectively, we have the following well-known formulas

$$\begin{aligned}
x_\omega &= \frac{2x_0'y_0'}{g} = \frac{v_0^2 \sin 2\alpha}{g}, \\
t_\omega &= \frac{2y_0'}{g} = \sqrt{\frac{2x_\omega \tan \alpha}{g}}, \\
\omega &= \alpha, \qquad x_\omega' = x_0', \qquad y_\omega' = -y_0', \\
t_s &= \frac{y_0'}{g} = \tfrac{1}{2}t_\omega, \\
y_s &= \frac{1}{2}\frac{y_0'^2}{g} = \tfrac{1}{2}gt_s^2 = \tfrac{1}{8}gt_\omega^2 = \tfrac{1}{4}x_\omega \tan \alpha, \\
\tau_s &= 0, \qquad x_s' = x_0', \qquad y_s' = 0.
\end{aligned} \tag{5}$$

Returning to the case of a resistance different from zero, we shall now derive qualitative properties of the trajectory under various hypotheses on the resistance. In § 2, hypothesis A, stating essentially that the resistance is positive, will be used to obtain a number of inequalities which all reduce to equalities of the type of (5) when the resistance is made equal to zero.* In § 3, the stronger hypothesis B, which states that the resistance depends on v and y alone, decreases or is stationary when y increases, and increases faster than v when the latter increases, is used to obtain additional inequalities of a similar character.

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* Under the special assumption that the resistance is positive and depends on v alone, so that $E = E(v) > 0$, some of these properties have been obtained before. Thus properties I, III, VI VII, X, the first inequality in XI, XII, XIV and XV were given by P. de Saint-Robert, Mem. Ac. Sc. Torino, ser. 2, vol. 16 (1855). Properties XIII and the part of XVI stating that the time of flight is less on the rising than on the falling branch, were obtained by Zaboudsky in his Exterior Ballistics (St. Petersburg, 1895).

Finally, the still stronger hypothesis C, to the effect that the resistance is the product of a function of v by a function of y with some further restrictions on the mode of increase of these functions, is introduced in § 4 for the purpose of investigating the existence of maxima and minima of the velocity v.*

2. **Positive resistance—Hypothesis A.** All inequalities in this paragraph will be derived from

HYPOTHESIS A: *For* $x \geqq 0$, $x' > 0$, $t \geqq 0$ *and all finite values of* y *and* y', *the function* $E = E(x, y, x', y', t)$ *is positive, and the derivatives* $\partial E/\partial x$, $\partial E/\partial y$, $\partial E/\partial x'$ *and* $\partial E/\partial y'$ *exist. Moreover, for any positive values of* a, b, c *and* d, *there exists an* $M = M(a, b, c, d)$ *such that the four derivatives and* E *itself are less in absolute value than* M *for* $0 \leqq x \leqq a$, $0 < x' \leqq b$, $-c \leqq y \leqq c$, $-d \leqq y' \leqq d$ *and* $t \geqq 0$ *but unrestricted upward.*

Since $x' > 0$ by equation (6) below, it follows from the general existence theorem for the solution of a system of differential equations, that the system (1) with the initial conditions (2) has then a unique solution which is defined (and finite) for any finite positive value of t.†

From the first of equations (1), we obtain

$$x' = x_0' e^{-\int_0^t E dt},$$

and since $E > 0$, it is seen at once that

I. *The horizontal velocity component* x' *is positive and decreases steadily from the value* x_0' *as* t *increases from zero.*

Since

$$x = \int_0^t x' dt \tag{7}$$

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* In the special case $E = E(v)$, the result that v has at most one minimum but no maximum was obtained by de Saint-Robert (l.c.).

† When E is not bounded, the trajectory may end in a point which is reached in a finite time, beyond which x and y are no longer real. To show this, let us transform (1) to the independent variable τ; we then obtain the following equations, which are derived in all textbooks on ballistics:

$$\frac{d(v \cos \tau)}{d\tau} = -\frac{v^2 E}{g}, \qquad \frac{dt}{d\tau} = -\frac{v}{g \cos \tau},$$

$$\frac{dx}{d\tau} = -\frac{v^2}{g}, \qquad \frac{dy}{d\tau} = -\frac{v^2 \tan \tau}{g}.$$

Now make $E = gv^{-3}$, which becomes infinite for $x' = y' = 0$; then the first equation is integrable immediately and gives

$$(v \cos \tau)^2 = (v_0 \cos \alpha)^2 - 2 \sin \alpha + 2 \sin \tau.$$

Assume $(v_0 \cos \alpha)^2 < 2 \sin \alpha$ and determine τ_1 by

$$\sin \tau_1 = \sin \alpha - \tfrac{1}{2}(v_0 \cos \alpha)^2;$$

then $v \cos \tau$ is imaginary for $\tau < \tau_1$, and the motion ends at the time t_1 determined by

$$t_1 = \frac{1}{g}\int_{\tau_1}^{\alpha} \sqrt{2(\sin \tau - \sin \tau_1)}\, \frac{d\tau}{\cos^2 \tau}.$$

and $x' > 0$ by I, it follows that

II. *The horizontal distance x increases steadily from zero as t increases from zero.*

Eliminating E between the two equations (1), we find

$$\frac{d}{dt}\left(\frac{y'}{x'}\right) = -\frac{g}{x'}, \tag{8}$$

and integrating,

$$\tan\tau - \tan\alpha = \frac{y'}{x'} - \frac{y_0'}{x_0'} = -g\int_0^t \frac{dt}{x'} = -g\int_0^x \frac{dx}{x'^2}, \tag{9}$$

whence, since $x' > 0$,

III. *The slope $\tan\tau$ decreases as t increases (or as x increases) so that the trajectory is concave downward.*

We may replace (9) by

$$\frac{dy}{dx} = \tan\alpha - g\int_0^x \frac{dx}{x'^2}; \tag{10}$$

since $x' < x_0'$ by I, we have,

$$\frac{dy}{dx} < \tan\alpha - g\int_0^x \frac{dx}{x_0'^2} = \tan\alpha - g\frac{x}{x_0'^2}$$

and integrating between the limits 0 and x,

$$y < \tan\alpha\cdot x - g\frac{x^2}{2x_0'^2}, \tag{11}$$

whence by comparison to (4),

IV. *The trajectory lies below the trajectory in vacuum corresponding to the same initial velocity components x_0' and y_0'.*

At an extreme of y, we have $y' = 0$, and consequently $y'' = -g < 0$ by (1), so that every extreme is a maximum. Therefore y, being less, by IV, than the maximum ordinate $y_0'^2/2g$ of the trajectory in vacuum, has a unique maximum and no minimum, so that

V. *The altitude y increases from $t = 0$ to its maximum y_s at $t = t_s$, and decreases steadily as t increases beyond t_s.*[*]

From the second of equations (1), it is seen that when $y' \geqq 0$, then $y'' < 0$ so that y' decreases as t increases, and since x' also decreases, the same is the case with $v = \sqrt{x'^2 + y'^2}$, or

VI. *The velocity and its vertical component decrease as t increases from 0 to t_s.*

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[*] In case $y_0' < 0$, it follows in the same way that y decreases steadily as t increases from zero.

Since x' decreases, (9) gives

$$\frac{y'}{x'} < \frac{y_0'}{x_0'} - g\int_0^t \frac{dt}{x_0'} = \frac{y_0' - gt}{x_0'},$$

and consequently

$$\lim_{t\to\infty} \frac{y'}{x'} = -\infty \tag{12}$$

or

VII. *The angle of slope* τ *decreases steadily toward* $-(\pi/2)$ *as* t *increases indefinitely.*

We shall now prove

VIII. *The trajectory intersects the x-axis for one and only one* $t = t_\omega$ *beyond* $t = t_s$, (*so that the range* x_ω *exists*).

There can be only one point of intersection, the trajectory being concave downward (III), and since $y > 0$ for $t = t_s$, this point of intersection exists if we can show that $y \to -\infty$ as $t \to \infty$. By I, x' decreases toward a limit $\geqq 0$ as $t \to \infty$, which leads us to distinguish three cases. First, assume that $\lim\limits_{t\to\infty} x' > 0$; then (12) shows that there exists a t_1 such that $y' < -1$ for $t > t_1$ and consequently $y - y_1 < -(t - t_1)$, or $y \to -\infty$ as $t \to \infty$.* Second, assume that $\lim\limits_{t\to\infty} x' = 0$, but that there exists an $\epsilon > 0$ and a t_1 such that $y' < -\epsilon$ when $t > t_1$; then $y - y_1 < -\epsilon(t - t_1)$ and $y \to -\infty$ as before.† Finally, assume that $x' \to 0$ and y is bounded downward as $t \to \infty$, but that y' approaches zero infinitely often, that is, for any $\epsilon > 0$ however small, the set of t-values defined by $-\epsilon \leqq y' < 0$ contains values greater than an arbitrarily chosen t_1. Since y is bounded, it follows from (11) that x is also bounded, and since $0 < x' < x_0'$, hypothesis A shows that there exists an M such that $|E| < M$ for all values of t for which $-1 \leqq y' < 0$. Now take for ϵ the smaller of the values 1 and $g/2M$; from the second equation (1), we obtain

$$y'' \leqq |Ey'| - g < M \cdot \frac{g}{2M} - g = -\tfrac{1}{2}g$$

for all t such that $-\epsilon \leqq y' < 0$. But y' obviously cannot decrease for all t in this set; for some point belonging to this set, we must therefore

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* Example: $E = x' - a$, where $0 < a < x_0'$. The first equation (1) gives at once

$$\frac{x' - a}{x'} = \frac{x_0' - a}{x_0'} e^{-at},$$

whence $x' \to a$ as $t \to \infty$.

† Example: $E = 1$. Equations (1) give $x' = x_0' e^{-t}$, $y' + g = (y_0' + g)e^{-t}$, so that $x' \to 0$ and $y' \to -g$ as $t \to \infty$.

have $y'' \geqq 0$, in contradiction to $y'' < -\frac{1}{2}g$. Hence the last case cannot occur, and our theorem is proved.*

In order to derive further inequalities, we shall use the following theorem which is due to Tchebychef:

When $u(x)$ and $v(x)$ are continuous for $a \leqq x \leqq b$, and both these functions increase (or both decrease) as x increases, then

$$(b-a)\int_a^b u(x)v(x)dx > \int_a^b u(x)dx \cdot \int_a^b v(x)dx,$$

but

$$(b-a)\int_a^b u(x)v(x)dx < \int_a^b u(x)dx \cdot \int_a^b v(x)dx$$

when one of the functions increases and the other decreases.†

Returning to our trajectory, we form the expression

$$\frac{d}{dx}(y - x\tan\tau) = \frac{1}{x'}\frac{d}{dt}\left(y - x\frac{y'}{x'}\right) = -\frac{x}{x'}\frac{d}{dt}\left(\frac{y'}{x'}\right) = g\frac{x}{x'^2},$$

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* That some assumption such as Hypothesis A is necessary to prove VIII is seen by the example $E = 3(1+t)^2$, where E increases indefinitely with t.

From equations (1),

$$x' = x_0'e^{1-(1+t)^3},$$

$$y' = e^{-(1+t)^3}\left[ey_0' - g\int_0^t e^{(1+u)^3}du\right].$$

When $0 < u < t$, we have $(1+u)^3 < (1+t)^2(1+u)$ and consequently

$$\int_0^t e^{(1+u)^3}du < \int_0^t e^{(1+t)^2(1+u)}du < \frac{e^{(1+t)^3}}{(1+t)^2},$$

so that

$$y' > y_0'e^{1-(1+t)^3} - \frac{g}{(1+t)^2},$$

$$y > y_0'\int_0^t e^{1-(1+t)^3}dt - g\left(1 - \frac{1}{1+t}\right).$$

For a sufficiently large y_0', it follows that $y > 0$ for large values of t, and consequently the trajectory does not intersect the x-axis (except at the origin).

† The following very simple proof was given by F. Franklin in the American Journal of Mathematics, vol. 7, p. 377 (1884): Consider the double integral

$$\int_a^b\int_a^b [u(t) - u(x)][v(t) - v(x)]dtdx = \int_a^b dt\int_a^b [u(t)v(t) - u(t)v(x) - u(x)v(t) + u(x)v(x)]dx$$

$$= (b-a)\int_a^b u(t)v(t)dt - \int_a^b u(t)dt \cdot \int_a^b v(x)dx - \int_a^b v(t)dt \cdot \int_a^b u(x)dx + (b-a)\int_a^b u(x)v(x)dx,$$

or replacing t by x in the single integrals to the right

$$\frac{1}{2}\int_a^b\int_a^b [u(t) - u(x)][v(t) - v(x)]dtdx = (b-a)\int_a^b u(x)v(x)dx - \int_a^b u(x)dx \cdot \int_a^b v(x)dx.$$

When $u(x)$ and $v(x)$ vary in the same sense, $u(t) - u(x)$ and $v(t) - v(x)$ have the same sign, so that their product is positive, and therefore also the double integral, which proves the first part of the theorem. The second part follows from the first upon replacing $u(x)$ by $-u(x)$.

the last step being a consequence of (8). Integrating from zero to x, we find

$$y = x \tan \tau + g \int_0^x \frac{x dx}{x'^2}. \tag{13}$$

In the integral, x increases and so does $1/x'^2$ by I; we may therefore apply Tchebychef's theorem and obtain

$$y > x \tan \tau + g \cdot \frac{1}{x} \cdot \int_0^x x dx \cdot \int_0^x \frac{dx}{x'^2}$$

$$= x \tan \tau + g \cdot \frac{1}{x} \cdot \frac{x^2}{2} \cdot - \frac{1}{g} (\tan \tau - \tan \alpha)$$

by (9), or reducing,

$$\frac{y}{x} > \tfrac{1}{2}(\tan \alpha + \tan \tau).$$

Now move the origin to the point x_1, y_1 on the trajectory; x, y and α are then replaced by $x - x_1$, $y - y_1$ and τ_1, and our inequality becomes

$$\frac{y - y_1}{x - x_1} > \tfrac{1}{2}(\tan \tau_1 + \tan \tau) \tag{14}$$

or

IX. *The arithmetic mean of the slopes at any two points on the trajectory is less than the slope of the chord joining these two points.*[*]

In the remainder of this paragraph, we shall denote by x_1, y_1 a point on the rising branch of the trajectory (where $y' > 0$), and by x_2, y_2 a point on the falling branch (where $y' < 0$), the corresponding times and velocity components being t_1, x_1', y_1' and t_2, x_2', y_2' respectively.

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[*] This is also readily proved by the trapezoid formula with remainder term for the evaluation of definite integrals:

$$\int_{x_1}^{x_2} f(x) dx = \frac{x_2 - x_1}{2} [f(x_1) + f(x_2)] - \frac{1}{2} \int_{x_1}^{x_2} (x - x_1)(x_2 - x) f''(x) dx$$

which is verified at once by integrating by parts in the last integral. Since $x - x_1$ and $x_2 - x$ are positive in the interval of integration, it follows that when $f''(x) < 0$ for $x_1 < x < x_2$, then

$$\int_{x_1}^{x_2} f(x) dx > \frac{x_2 - x_1}{2} [f(x_1) + f(x_2)].$$

Now let $f(x) = \tan \tau$; then $f'(x) = - g/x'^2$ by (8), and

$$f''(x) = \frac{1}{x'} \frac{d}{dt} (- g/x'^2) = 2gx''/x'^4 = - 2gE/x'^3$$

by the first of (1). Since $E > 0$, $x' > 0$, we have $f''(x) < 0$ and the last inequality becomes

$$y_2 - y_1 = \int_{x_1}^{x_2} \tan \tau \, dx > \frac{x_2 - x_1}{2} [\tan \tau_1 + \tan \tau_2],$$

which is identical with (14).

Making $x = x_2$, $y = y_2 = y_1$ in (14), we obtain $\tan \tau_1 + \tan \tau_2 < 0$, or

$$-\tau_2 > \tau_1, \tag{15}$$

and in particular, for $y_1 = y_2 = 0$,

$$\omega > \alpha, \tag{16}$$

whence

X. *At two points of equal altitude the slope of the trajectory is numerically greater on the falling than on the rising branch; in particular, the angle of fall ω is greater than the angle of departure α.*

Next, make $x_1 = y_1 = 0$, $x = x_s$ and $y = y_s$ in (14), whence $y_s/x_s > \frac{1}{2} \tan \alpha$, and furthermore $x_1 = x_s$, $y_1 = y_s$, $x = x_\omega$ and $y = 0$, whence $-y_s/(x_\omega - x_s) > \frac{1}{2} \tan \omega$, so that, combining these two inequalities,

$$\tfrac{1}{2} x_s \tan \alpha < y_s < \tfrac{1}{2}(x_\omega - x_s) \tan \omega. \tag{17}$$

Since $\tan \tau = dy/dx$, we have

$$x_1 = \int_0^{y_1} \frac{dy_1}{\tan \tau_1}, \qquad x_\omega - x_2 = \int_0^{y_2} \frac{dy_2}{-\tan \tau_2},$$

$$x_s - x_1 = \int_{y_1}^{y_s} \frac{dy_1}{\tan \tau_1}, \qquad x_2 - x_s = \int_{y_2}^{y_s} \frac{dy_2}{-\tan \tau_2},$$

and for $y_1 = y_2$, by means of (15),

$$x_1 > x_\omega - x_2, \qquad x_s - x_1 \geqq x_2 - x_s, \tag{18}$$

where the equality sign holds only when $x_1 = x_s = x_2$. In particular, for $x_1 = x_s = x_2$, (18) gives $x_s > \frac{1}{2} x_\omega$, and combining this with the outer terms in (17), we find

XI. *The abscissa x_s of the summit of the trajectory satisfies the inequalities*

$$\tfrac{1}{2} x_\omega < x_s < \frac{\tan \omega}{\tan \alpha + \tan \omega} \cdot x_\omega. \tag{19}$$

Introducing $x_s > \frac{1}{2} x_\omega$ in (17), we find

$$\tfrac{1}{4} x \tan \alpha_\omega < y_s < \tfrac{1}{4} x_\omega \tan \omega, \tag{20}$$

or

XII. *The maximum ordinate y_s lies between the maximum ordinates of the two trajectories in vacuum having the same range x_ω and the angles of departure α and ω respectively.**

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* For another proof based on Rolle's theorem, see Charbonnier, Balistique extérieure rationnelle, vol. 1 (Paris, Doin, 1907), p. 193.

Upon multiplication by $2y'$, the second of equations (1) may be written $(y'^2)' = -2Ey'^2 - 2gy'$ or, since $dy/dt = y'$,

$$\frac{d}{dy} y'^2 = -2Ey' - 2g.$$

Integrating from y to y_s we find, since $y_s' = 0$,

$$y'^2 = \int_y^{y_s} 2Ey'dy + 2g(y_s - y).$$

Since y' is positive on the rising and negative on the falling branch, it follows that

$$y_1'^2 > 2g(y_s - y_1), \qquad y_2'^2 < 2g(y_s - y_2),$$

which may also be written, observing that y_2' is negative,

$$(21) \qquad y_1' > \sqrt{2g(y_s - y_1)}, \qquad -y_2' < \sqrt{2g(y_s - y_2)}.$$

When $y_1 = y_2$, (21) gives

$$(22) \qquad y_1' > -y_2'$$

or

XIII. *At two points of equal altitude on the trajectory, the vertical velocity component is numerically greater on the rising than on the falling branch.*

Since also $x_1' > x_2'$ by I, it follows from (22) that for $y_1 = y_2$

$$(23) \qquad v_1 > v_2,$$

or

XIV. *At two points of equal altitude on the trajectory, the velocity is greater on the rising than on the falling branch.*

For the arc s of the trajectory, we have $ds/dy = 1/\sin \tau$; hence, denoting by s_ω the total length of arc,

$$s_1 = \int_0^{y_1} \frac{dy_1}{\sin \tau_1}, \qquad s_\omega - s_2 = \int_0^{y_2} \frac{dy_2}{-\sin \tau_2},$$

and by (15), for $y_1 = y_2$,

$$(24) \qquad s_1 > s_\omega - s_2.$$

In particular, it follows from (24) that

XV. *The length of arc of the rising branch of the trajectory is greater than that of the falling branch.*

On account of $dt/dy = 1/y'$, we have

$$t_s - t_1 = \int_{y_1}^{y_s} \frac{dy_1}{y_1'}, \qquad t_2 - t_s = \int_{y_2}^{y_s} \frac{dy_2}{-y_2'},$$

and by (21)

$$t_s - t_1 < \int_{y_1}^{y_s} \frac{dy_1}{\sqrt{2g(y_s - y_1)}}, \qquad t_2 - t_s > \int_{y_2}^{y_s} \frac{dy_2}{\sqrt{2g(y_s - y_2)}},$$

or performing the integrations

$$(25) \qquad t_s - t_1 < \sqrt{\frac{2(y_s - y_1)}{g}}, \qquad t_2 - t_s > \sqrt{\frac{2(y_s - y_2)}{g}}.$$

In particular, for $y_1 = y_2 = 0$

$$(26) \qquad t_s < \sqrt{\frac{2y_s}{g}} < t_\omega - t_s,$$

or

XVI. *The time of flight on the rising branch is less, and that on the falling branch greater, than the corresponding time of flight on a trajectory in vacuum with the same maximum ordinate.*

It should be noted in this connection that the empirical inequality

$$y_s > \tfrac{1}{8} g t_\omega^2$$

proposed by several ballisticians, cannot be universally true. On the contrary, the inequality sign must be reversed when the angle of departure is sufficiently small, as will be shown on another occasion, in connection with the use of power series in exterior ballistics.

To obtain inequalities connecting t_ω with x_ω, we write

$$t_2 - t_1 = \int_{t_1}^{t_2} dt = \int_{t_1}^{t_2} x' \frac{dt}{x'}$$

and apply Tchebychef's theorem:

$$t_2 - t_1 < \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x' dt \cdot \int_{t_1}^{t_2} \frac{dt}{x'} = \frac{1}{t_2 - t_1} (x_2 - x_1) \cdot \frac{1}{g} (\tan \tau_1 - \tan \tau_2)$$

whence

$$(27) \qquad t_2 - t_1 < \sqrt{\frac{x_2 - x_1}{g} (\tan \tau_1 - \tan \tau_2)}.*$$

Applying (27) to the rising and falling branches of the trajectory, we find

$$t_s < \sqrt{\frac{x_s \tan \alpha}{g}}, \qquad t_\omega - t_s < \sqrt{\frac{(x_\omega - x_s) \tan \omega}{g}},$$

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* Another proof of (27) is obtained from

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{x'}$$

by the application of Schwarz' inequality:

$$(t_2 - t_1)^2 < \int_{x_1}^{x_2} dx \, . \int_{x_1}^{x_2} \frac{dx}{x'^2} = (x_2 - x_1) \, . \, \frac{1}{g} (\tan \tau_1 - \tan \tau_2).$$

and by addition

$$t_\omega < \sqrt{\frac{x_s \tan\alpha}{g}} + \sqrt{\frac{(x_\omega - x_s)\tan\omega}{g}}. \tag{28}$$

Replacing x_s by $\frac{1}{2}x_\omega$ increases the expression to the right; in fact, it follows from $x_s > \frac{1}{2}x_\omega$ and $\tan\alpha < \tan\omega$ that

$$\begin{aligned}\sqrt{x_s \tan\alpha} + \sqrt{(x_\omega - x_s)\tan\omega} - \left(\sqrt{\tfrac{1}{2}x_\omega \tan\alpha} + \sqrt{\tfrac{1}{2}x_\omega \tan\omega}\right)\\ = \left(\sqrt{x_s} - \sqrt{\tfrac{1}{2}x_\omega}\right)\sqrt{\tan\alpha} - \left(\sqrt{\tfrac{1}{2}x_\omega} - \sqrt{x_\omega - x_s}\right)\sqrt{\tan\omega}\\ < \sqrt{\tan\alpha}\left(\sqrt{x_s} + \sqrt{x_\omega - x_s} - 2\sqrt{\tfrac{1}{2}x_\omega}\right),\end{aligned}$$

and the last factor is negative by the algebraic identity

$$\begin{aligned}\left(\sqrt{x_s} + \sqrt{x_\omega - x_s} + \sqrt{2x_\omega}\right)\left(\sqrt{x_s} + \sqrt{x_\omega - x_s} - \sqrt{2x_\omega}\right)\\ = -\left(\sqrt{x_s} - \sqrt{x_\omega - x_s}\right)^2.\end{aligned}$$

Therefore (28) may be replaced by

$$t_\omega < \sqrt{\frac{x_\omega}{2g}}\left(\sqrt{\tan\alpha} + \sqrt{\tan\omega}\right).$$

On the other hand, we have since x' decreases,

$$\tan\alpha = g\int_0^{t_s}\frac{dt}{x'} < g\frac{t_s}{x_s'},$$

$$x_\omega - x_s = \int_{t_s}^{t_\omega} x'dt < x_s'(t_\omega - t_s),$$

and multiplying these two inequalities

$$t_s(t_\omega - t_s) > \frac{x_\omega - x_s}{g}\tan\alpha.$$

But $t_s(t_\omega - t_s) < \frac{1}{4}t_\omega^2$, and $x_\omega - x_s > x_\omega \tan\alpha/(\tan\alpha + \tan\omega)$ by (19), so that

$$\tfrac{1}{4}t_\omega^2 > \frac{x_\omega \tan^2\alpha}{g(\tan\alpha + \tan\omega)},$$

and consequently we find that

XVII. *The time of flight is bounded in terms of the range by the inequalities*

$$\sqrt{\frac{x_\omega}{2g}}\cdot\sqrt{\frac{8\tan^2\alpha}{\tan\alpha + \tan\omega}} < t_\omega < \sqrt{\frac{x_\omega}{2g}}\left(\sqrt{\tan\alpha} + \sqrt{\tan\omega}\right). \tag{29}$$

3. Resistance depending on v and y only. Hypothesis B. All properties in this paragraph will be derived from the following hypothesis (except XIX, which requires the stronger assumption stated):

HYPOTHESIS B: *The function* $E = E(v, y)$ *depends on* v *and* y *alone. For* $v > 0$ *and all finite values of* y *it is positive and has derivatives* $\partial E/\partial v$ *and* $\partial E/\partial y$, *and for any positive values of* a *and* c, *there exists an* $M = M(a, c)$ *such that the two derivatives and* E *itself are less in absolute value than* M *for* $0 < v \leqq a,\ -c \leqq y \leqq c$. *Moreover,*

$$\frac{\partial E}{\partial v} > 0, \qquad \frac{\partial E}{\partial y} \leqq 0,$$

so that E *increases when* v *increases, but does not increase when* y *increases.*

We shall first derive some inequalities involving the horizontal velocity components at the origin, the summit and the point of fall. From the first of equations (1), we obtain

$$x_0' - x_s' = \int_0^{t_s} Ex'dt = \int_0^{y_s} \frac{E(v_1, y_1)}{\tan \tau_1} dy_1,$$

and similarly

$$x_s' - x_\omega' = \int_0^{y_s} \frac{E(v_2, y_2)}{-\tan \tau_2} dy_2.$$

For $y_1 = y_2$, we have $\tan \tau_1 < -\tan \tau_2$ by X, and $v_1 > v_2$ by XIV, so that $E(v_1, y_1) > E(v_2, y_2)$ by hypothesis B, and consequently $x_0' - x_s' > x_s' - x_\omega'$ or

XVIII. *The horizontal velocity component at the summit is less than the arithmetic mean of its values at the origin and at the point of fall.*

$$x_s' < \tfrac{1}{2}(x_0' + x_\omega'). \tag{30}$$

From the first of equations (1), it follows that

$$\log \frac{x_0'}{x_s'} = \int_0^{t_s} -\frac{x''}{x'} dt = \int_0^{t_s} E dt = \int_0^{y_s} \frac{E(v_1, y_1)}{y_1'} dy_1 = \int_0^{y_s} \frac{E(v_1, y_1)}{v_1} \frac{dy_1}{\sin \tau_1},$$

and similarly

$$\log \frac{x_s'}{x_\omega'} = \int_0^{y_s} \frac{E(v_2, y_2)}{v_2} \frac{dy_2}{-\sin \tau_2}.$$

If, for $y_1 = y_2$ (whence $v_1 > v_2$), we have $E(v_1, y_1)/v_1 \geqq E(v_2, y_2)/v_2$, it follows that $\log (x_0'/x_s') > \log (x_s'/x_\omega')$, or

XIX. *When, in addition to hypothesis* B, $E(v, y)$ *is such that* $E(v, y)/v$ *does not decrease when* v *increases and* y *remains constant, the horizontal velocity component at the summit is less than the geometric mean of its values at the origin and at the point of fall:*

$$x_s' < \sqrt{x_0' x_\omega'}. \tag{31}$$

In the equation

$$x_0' - x_s' = \int_0^{t_s} Ex'dt,$$

x' decreases, and E also decreases by hypothesis B, since v decreases (VI) and y increases; hence Tchebychef's theorem gives

$$x_0' - x_s' > \frac{1}{t_s} \cdot \int_0^{t_s} x'dt \cdot \int_0^{t_s} Edt = \frac{x_s}{t_s} \log\left(\frac{x_0'}{x_s'}\right),$$

or

$$x_s < t_s \frac{x_0' - x_s'}{\log x_0' - \log x_s'}. \tag{32}$$

Similarly the equation

$$\frac{1}{x_s'} - \frac{1}{x_0'} = \int_0^{t_s} - \frac{x''}{x'^2} dt = \int_0^{t_s} E \frac{dt}{x'}$$

gives

$$\frac{1}{x_s'} - \frac{1}{x_0'} < \frac{1}{t_s} \cdot \log\left(\frac{x_0'}{x_s'}\right) \cdot \frac{\tan \alpha}{g}$$

or

$$t_s < x_0' x_s' \frac{\log x_0' - \log x_s'}{x_0' - x_s'} \frac{\tan \alpha}{g} \tag{33}$$

whence by (32)

$$x_s < x_0' x_s' \frac{\tan \alpha}{g} \tag{34}$$

and since $x_s > \frac{1}{2} x_\omega$ by (19), we have

$$x_s' > \frac{g x_\omega \cot \alpha}{2 x_0'}. \tag{35}$$

Further inequalities of this type may be obtained, but they are too complicated to be of much interest.

We now proceed to derive some properties of the trajectory when t increases indefinitely.

XX.. *For all values of t, the velocity is bounded by*

$$v \leqq \max\left(v_0, \frac{g}{E(v_0, 0)}\right). \tag{36}$$

Since $vv' = x'x'' + y'y''$, we obtain from (1)

$$v' = - vE - g\frac{y'}{v}. \tag{37}$$

By VI and XIV, $v < v_0$ for $0 < t \leqq t_\omega$, and we need therefore only show that the upper bound (36) cannot be reached for $t > t_\omega$. Assume t_1 to be the smallest value of t beyond t_ω for which the upper bound (36) is reached; since $v_\omega < v_0$, it follows that v must either increase through t_1 or have a maximum there, so that $v' \geqq 0$ at t_1. But at t_1, we have $v \geqq v_0$ and $y < 0$, hence $E(v, y) \geqq E(v_0, 0)$ by hypothesis B, and

$$- y'/v = - \sin \tau < 1$$

by VII, so that (37) gives $v' < g - vE(v_0, 0)$. Since we have assumed that $v \geqq g/E(v_0, 0)$ at t_1, it follows that $v' < 0$ for this value of t, and this contradiction to $v' \geqq 0$ proves our theorem.

Equation (8) may be written

$$\frac{d\tau}{dt} = - \frac{g}{x'} \cos^2 \tau = - g \frac{x'}{v^2},$$

and therefore $dx/d\tau = - v^2/g$ or

$$x = \frac{1}{g} \int_\tau^\alpha v^2 d\tau.$$

Now v^2 is bounded by XX, and $\tau > - \pi/2$; consequently x is bounded and being an increasing function of t, x tends towards the limit given by making $\tau = - \pi/2$ in the above integral. Hence

XXI. *The falling branch of the trajectory has the vertical asymptote*

$$x = \frac{1}{g} \int_{-(\pi/2)}^\alpha v^2 d\tau.$$

Since $x = \int_0^t x' d\tau$, and x' is positive and decreasing, it follows from the boundedness of x that

$$x' \to 0 \text{ as } t \to \infty.$$

4. **Maxima and minima of the velocity. Hypothesis C.** It is the main purpose of this paragraph to show that, under fairly weak assumptions, there exist no extremes of v beyond a certain value of t. If there are infinitely many extremes, the points at which they occur must therefore have all their limiting points at finite distance, and by a stronger assumption, this possibility may also be excluded, so that v has then only a finite number of extremes.

The first step in this investigation consists in showing that $vE \to g$ as $t \to \infty$. Suppose first that there exist a t_1 such that for $t > t_1$, vE either increases or decreases steadily as t increases. Since $vE > 0$, it then follows that vE either increases indefinitely or tends toward a limit greater than or equal to zero. Now $- y'/v = \sin(- \tau) \to 1$ by VII, and unless $vE \to g$, it then follows from (37) that there exist an $\epsilon > 0$ and a t_2 such that either $v' > \epsilon$ for $t > t_2$, or $v' < - \epsilon$ for $t > t_2$, according as $\lim vE < g$ or $\lim vE > g$ (including $vE \to \infty$). Consequently we have either

$v > v_2 + \epsilon(t - t_2) \to \infty$ as $t \to \infty$, in contradiction to XX, or $v < v_2 - \epsilon(t - t_2) \to -\infty$ in contradiction to $v \geqq 0$. Hence $vE \to g$ under the assumption made.

There remains to be considered the case when extremes of vE occur for indefinitely increasing values of t. It is obvious that in this case, lim inf vE as $t \to \infty$ equals the inferior limit of the minima of vE, and lim sup vE as $t \to \infty$ equals the superior limit of the maxima. Now

$$\frac{d(vE)}{dt} = vE\left[\left(\frac{1}{v} + \frac{1}{E}\frac{\partial E}{\partial v}\right)v' + \frac{1}{E}\frac{\partial E}{\partial y}y'\right]$$

$$= vE\left[-\left(\frac{1}{v} + \frac{1}{E}\frac{\partial E}{\partial v}\right)\left(vE + g\frac{y'}{v}\right) + \frac{1}{E}\frac{\partial E}{\partial y}y'\right]$$

by (37), so that at an extreme of vE

$$\left(\frac{1}{v} + \frac{1}{E}\frac{\partial E}{\partial v}\right)\left(vE + g\frac{y'}{v}\right) = -\frac{1}{E}\frac{\partial E}{\partial y}\cdot - y'.$$

Since $\partial E/\partial v > 0$, $\partial E/\partial y \leqq 0$ by hypothesis B, it follows that for $y' \geqq 0$, the expression to the left is positive and that to the right negative or zero, so that the extremes of vE occur for $y' < 0$. The first factor to the left and the right hand member being then both positive, it follows that $vE + gy'/v > 0$, and since $\partial E/\partial v > 0$, we have

$$\frac{1}{v}\left(vE + g\frac{y'}{v}\right) < -\frac{1}{E}\frac{\partial E}{\partial y}\cdot - y';$$

moreover, $0 < -y' < v$, so that finally

$$-g\frac{y'}{v} < vE < -g\frac{y'}{v} - \frac{1}{E}\frac{\partial E}{\partial y}\cdot v^2 \tag{38}$$

at an extreme of vE.

On account of $-y'/v \to 1$ as $t \to \infty$, it follows from (38) that under hypothesis B

$$\liminf_{t \to \infty} vE \geqq g. \tag{39}$$

Before proceeding further, we shall prove the following

Lemma. Let $y(x)$ *and* $z(x)$ *be solutions of the differential equations*

$$\frac{dy}{dx} = f(x, y), \qquad \frac{dz}{dx} = \varphi(x, z)$$

with the initial conditions $y = z = y_0$ *for* $x = x_0$, *and assume that* $f(x, u)$ *and* $\varphi(x, u)$ *are continuous for* $x \geqq x_0$ *and* $|u - y(x)| < \epsilon$ *(or for* $x \geqq x_0$ *and* $|u - z(x)| < \epsilon$*), and that moreover* $f(x, u) < \varphi(x, u)$ *for* $x \geqq x_0$ *and*

$u = y(x)$ *(or for* $x \geqq x_0$ *and* $u = z(x)$*). Then*

$$y(x) < z(x)$$

for $x > x_0$.

Since, by hypothesis, $f(x, u)$ and $\varphi(x, u)$ are continuous at $x = x_0$, $u = y_0$, and $f(x_0, y_0) - \varphi(x_0, y_0) < 0$, it follows that $f(x, y(x)) - \varphi(x, z(x)) < 0$ for $x_0 \leqq x \leqq x_0 + \delta$, when δ is sufficiently small, so that $dy/dx < dz/dx$ and consequently $y < z$ for $x_0 < x < x_0 + \delta$. Now assume that $y(x) < z(x)$ for $x_0 < x < x_1$, but $y(x_1) = z(x_1)$. Then the same argument as before shows that $y(x) < z(x)$ for $x_1 < x < x_1 + \delta$, where δ is sufficiently small, and writing $v(x) = y(x) - z(x)$, we have for h positive and sufficiently small, $v(x_1 - h) < 0$, $v(x_1) = 0$, $v(x_1 + h) < 0$, so that

$$\frac{v(x_1 - h) - v(x_1)}{-h} > 0 > \frac{v(x_1 + h) - v(x_1)}{h},$$

and for $h \rightarrow 0$, we find $dv/dx = 0$ for $x = x_1$. But we have

$$\begin{aligned}\left(\frac{dv}{dx}\right)_{x=x_1} &= f(x_1, y(x_1)) - \varphi(x_1, z(x_1)) \\ &= f(x_1, y(x_1)) - \varphi(x_1, y(x_1)) < 0\end{aligned}$$

by hypothesis, and this contradiction shows that there is no $x_1 > x_0$ for which $y(x_1) = z(x_1)$, so that $y(x) < z(x)$ for $x > x_0$.

As an application, write $\eta = -y$, so that $\eta \rightarrow \infty$ as $t \rightarrow \infty$, and compare the equation

$$(40) \qquad \frac{dv}{d\eta} = \frac{g}{v} - \frac{v}{\eta'} E(v, -\eta)$$

obtained from (37) by a change of variable, with the equation

$$(41) \qquad \frac{dV}{d\eta} = \frac{g}{V} - E(V, -\eta),$$

the initial conditions being $v = V = v_1$ for $\eta = \eta_1$ and η_1 belonging to a point on the falling branch of the trajectory. Since $0 < -y' = \eta' < v$, we have

$$\frac{g}{v} - \frac{v'}{\eta} E(v, -\eta) < \frac{g}{v} - E(v, -\eta),$$

and by the lemma, it follows that

$$(42) \qquad v(\eta) < V(\eta) \qquad \text{for} \qquad \eta > \eta_1.$$

The mechanical interpretation of this is obvious, since (41) is the equation of motion of a particle projected vertically downward.

Now suppose that V has a minimum $= V_2$ for $\eta = \eta_2$; then, by (41),

$V_2E(V_2, -\eta_2) = g$. For $\eta_2 < \eta < \eta_2 + \delta$ and δ sufficiently small, we have $V > V_2$ since V_2 is a minimum, and by hypothesis B, it follows that $VE(V, -\eta) > V_2E(V_2, -\eta) \geqq V_2E(V_2, -\eta_2) = g$, so that, by (41), $dV/d\eta < 0$ or V decreases, which contradicts the assumption of a minimum. Hence V has at most one extreme, which is a maximum, and therefore, as $t \rightarrow \infty$ or $\eta \rightarrow \infty$, V tends either to a finite limit or to infinity, so that η sufficiently large, $V \geqq c$ where $c \geqq 0$. Now assume that $E(c, y) \rightarrow \infty$ as $y \rightarrow -\infty$ for any positive value of the constant c; then if $V \geqq c > 0$, it follows from (41) that $dV/d\eta \rightarrow -\infty$ as $y \rightarrow -\infty$ and consequently $V \rightarrow -\infty$ which is impossible. Consequently $V \rightarrow 0$ as $t \rightarrow \infty$ and a fortiori $v \rightarrow 0$ by (42), or

XXII. *When in addition to hypothesis B, we assume that $E(c, y) \rightarrow \infty$ as $y \rightarrow -\infty$ for any positive value however small of the constant c, then*

$$v \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Returning to the proof of $vE \rightarrow g$, we now introduce

HYPOTHESIS C. *The function $E(v, y)$ has the form*

$$E(v, y) = G(v) \cdot H(y),$$

where $G(v) > 0$ and $G'(v)$ exists and is > 0 for $v > 0$, and is bounded for $0 < v < a$, where a is as large as we please; moreover $H(y) > 0$, $H(y) \rightarrow \infty$ as $y \rightarrow -\infty$, $H'(y)$ exists and is negative and bounded for $-b < y < b$, where b is as large as we please, and $h(y) = -d\,[\log H(y)]/dy$ satisfies the condition

$$e^{-cy}h(y) \rightarrow \infty \text{ as } y \rightarrow -\infty$$

for any positive constant c however small. Finally, any one of the three following assumptions is made:

1) *$h(y)$ is bounded as $y \rightarrow -\infty$; no additional condition on $G(v)$.*

2) $\dfrac{h(y)}{H(y)} \rightarrow 0$ *as $y \rightarrow -\infty$, and $G(0) = 0$.*

3) *For some constant $m \geqq 1$, and some positive constant c_1,*

$$\frac{h(y)}{[H(y)]^{2/(m+1)}} \rightarrow 0 \text{ as } y \rightarrow -\infty,$$

and $G(v) > c_1v^m$ for v sufficiently small.

This form of $E(v, y)$ is the one used in actual computations, $G(v)$ being given empirically in form of a table and assumed to satisfy the conditions $C1$. The factor $H(y)$ is introduced to account for the decrease in air resistance due to the decrease of the density of the air with increasing altitude; the expression formerly used $H(y) = (1 - ky)^n$, where k and

n are positive constants, has been replaced recently by $H(y) = e^{-hy}$, where h is a positive constant. It is seen at once that both these expressions for $H(y)$ satisfy $C1$.

Under hypothesis $C1$, it follows from (38) and XXII that for the maxima of vE

$$vE < -g\frac{y'}{v} + h(y)\cdot v^2 \rightarrow g \text{ as } t \rightarrow \infty,$$

so that $\limsup vE \leqq g$ as $t \rightarrow \infty$, and together with (39) this gives $vE \rightarrow g$.

Under hypothesis C2, it is sufficient to prove $\limsup VE(V, -\eta) \leqq g$ as $\eta \rightarrow \infty$, since for the same value of $\eta > \eta_1$, $v < V$ by (42) and consequently $vE(v, -\eta) < VE(V, -\eta)$.

From (41) we obtain

$$\frac{d\,VG(V)}{d\eta} = \frac{G(V) + VG'(V)}{V}[g - VG(V)\,H(-\eta)]. \tag{43}$$

Since $G(0) = 0$, we have $G(V) = G(V) - G(0) = VG'(\theta V)$, $0 < \theta < 1$, and since G' is bounded (V being bounded, tending toward zero as $\eta \rightarrow \infty$), it follows that

$$0 < \frac{G(V) + VG'(V)}{V} < k,$$

where k is a constant. With the equation of comparison

$$\frac{d\xi}{d\eta} = k[g - \xi H(-\eta)] \tag{44}$$

and the initial conditions $\xi = VG(V)$ for $\eta = \eta_1$, it follows from the lemma that $VG(V) < \xi$ for $\eta > \eta_1$, so that all we need to prove is

$$\limsup_{\eta \rightarrow \infty} H(-\eta)\cdot\xi \leqq g. \tag{45}$$

The solution of (44) taking the value ξ_1 for $\eta = \eta_1$ is

$$\xi = e^{-\int_{\eta_1}^{\eta} kH\,d\eta}\left[\xi_1 + g\int_{\eta_1}^{\eta_2} ke^{\int_{\eta_1}^{\eta} kH\,d\eta}d\eta + g\int_{\eta_2}^{\eta} ke^{\int_{\eta_1}^{\eta} kH\,d\eta}d\eta\right],$$

where η_2 is any constant greater than η_1, and $H = H(-\eta)$. Integrating by parts in the last integral, we find

$$\begin{aligned}\int_{\eta_2}^{\eta} ke^{\int_{\eta_1}^{\eta} kH\,d\eta}d\eta &= \int_{\eta_2}^{\eta} \frac{1}{H(-\eta)}\frac{d}{d\eta}[e^{\int_{\eta_1}^{\eta} kH\,d\eta}]d\eta \\ &= \left[\frac{1}{H}e^{\int_{\eta_1}^{\eta} kH\,d\eta}\right]_{\eta_2}^{\eta} + \int_{\eta_2}^{\eta}\frac{h(-\eta)}{H(-\eta)}e^{\int_{\eta_1}^{\eta} kH\,d\eta}d\eta,\end{aligned}$$

and consequently

$$\begin{aligned}H(-\eta)\cdot\xi - g &= gHe^{-\int_{\eta_1}^{\eta} kH d\eta}\int_{\eta_2}^{\eta}\frac{h}{H}e^{\int_{\eta_1}^{\eta} kH d\eta}d\eta \\ &+ He^{-\int_{\eta_1}^{\eta} kH d\eta}\left[\xi_1 + g\int_{\eta_1}^{\eta_2} ke^{\int_{\eta_1}^{\eta} kH d\eta}d\eta - \frac{g}{H(-\eta_2)}e^{\int_{\eta_1}^{\eta_2} kH d\eta}\right],\end{aligned} \tag{46}$$

For $\eta > \eta_2$, the first term to the right is less than

$$\frac{g\int_{\eta_2}^{\eta}\frac{h}{H}e^{\int_{\eta_1}^{\eta} kH d\eta}d\eta}{\frac{1}{H(-\eta)}e^{\int_{\eta_1}^{\eta} kH d\eta} - \frac{1}{H(-\eta_2)}e^{\int_{\eta_1}^{\eta_2} kH d\eta}},$$

and by the well-known formula

$$\frac{\varphi(\eta) - \varphi(\eta_2)}{\psi(\eta) - \psi(\eta_2)} = \frac{\varphi'(\eta_3)}{\psi'(\eta_3)},$$

where η_3 is some value between η_2 and η, this expression equals

$$\frac{g\frac{h(-\eta_3)}{H(-\eta_3)}}{k - \frac{h(-\eta_3)}{H(-\eta_3)}},$$

and since $\eta_3 > \eta_2$, it follows from hypothesis C2 that this expression may be made less than any $\epsilon > 0$ however small by taking η_2 sufficiently large. Having thus fixed η_2, the second term to the right in (46) approaches zero as $\eta \to \infty$, since

$$He^{-\int_{\eta_1}^{\eta} kH d\eta} \to 0 \text{ as } \eta \to \infty,$$

the logarithmic derivative of this expression being

$$H(-\eta)\left[\frac{h(-\eta)}{H(-\eta)} - k\right],$$

which approaches $-\infty$ when $\eta \to \infty$ by hypothesis C2. Consequently, (46) gives

$$\limsup H(-\eta)\cdot\xi - g \leqq \epsilon,$$

which is equivalent to (45), and hence $vE \to g$ as $t \to \infty$.

Under hypothesis C3, we have

$$\frac{g}{V} - G(V)\,H(-\eta) < \frac{g}{V} - c_1 V^m H(-\eta),$$

and comparing V defined by (41), to W defined by

$$\frac{dW}{d\eta} = \frac{g}{W} - c_1 W^m H(-\eta) \tag{47}$$

and the initial condition $W = V = v_1$ for $\eta = \eta_1$, we have $v < V < W$ for $\eta > \eta_1$. But in (47), we have $G(W) = c_1 W^m$ so that $G(0) = 0$, and hypothesis C3 is stronger in respect to H than C2; hence we have, from what has been proved under hypothesis C2,

$$\limsup_{\eta \to \infty} c_1 W^{m+1} H(-\eta) \leqq g,$$

and consequently, for η sufficiently large

$$c_1 W^{m+1} H(-\eta) < 2g.$$

Therefore

$$v^2 h(y) < V^2 h(y) < W^2 h(y) < \left(\frac{2g}{c_1}\right)^{2/(m+1)} \frac{h(y)}{(H(y))^{2/(m+1)}} \to 0$$

as $y \to -\infty$ by C3, and from (38) we conclude that the upper limit of the maxima of vE does not exceed g. Hence $vE \to g$ under hypothesis C3, so that we have the theorem:

XXIII. *Under hypothesis* C, *the retardation tends toward the limit* g *on the falling branch of the trajectory*:

$$vE(v, y) \to g \text{ as } t \to \infty. \tag{48}$$

At an extreme of v, we have $v' = 0$, or

$$vE = -g\frac{y'}{v} \tag{49}$$

by (37), and

$$\begin{aligned} v'' &= -\frac{\partial}{\partial v}\left(vE + g\frac{y'}{v}\right) \cdot v' - vE\frac{\partial \log E}{\partial y} y' - g\frac{y''}{v} \\ &= -\frac{\partial}{\partial v}\left(vE + g\frac{y'}{v}\right) \cdot v' + vEh(y) \cdot y' + \frac{g}{v}(Ey' + g); \end{aligned}$$

eliminating vE by (49), we find that for $v' = 0$,

$$v'' = \frac{g}{v^3}(gx'^2 - v^2 y'^2 h(y)). \tag{50}$$

For a minimum of v, we have $v'' \geqq 0$, and since $h(y) > 0$ by hypothesis C and $v^2 > y'^2$, it follows from (50) that

$$h(y)\left(\frac{y'^2}{x'}\right)^2 < g. \tag{51}$$

From equations (1), it is readily seen that

$$\frac{d}{dt}\log\frac{y'^2}{x'} = -\frac{1}{y'}(2g + Ey'),$$

or, since $\eta = -y$,

$$\frac{d}{d\eta}\log\frac{y'^2}{x'} = \frac{1}{y'^2}(2g + Ey').$$

Since $Ey' \rightarrow -Ev \rightarrow -g$ by (48), and $y'^2 < v^2$ is bounded by XX, it follows that there exists a positive constant c such that, for η_1 sufficiently large,

$$\frac{d}{d\eta}\log\frac{y'^2}{x'} > \frac{c}{2} \qquad \text{for} \qquad \eta \geqq \eta_1,$$

and consequently

$$\frac{y'^2}{x'} > \left(\frac{y'^2}{x'}\right)_1 e^{c(\eta-\eta_1)/2}.$$

Since $e^{c\eta}h(-\eta) \rightarrow \infty$ as $\eta \rightarrow \infty$ by hypothesis C, it follows that

$$h(y)\left(\frac{y'^2}{x'}\right)^2 \rightarrow \infty \qquad \text{as} \qquad t \rightarrow \infty,$$

and comparing this to (51), it is seen that for sufficienlty large values of t, no minima of v can occur, and hence no maxima, since these alternate with the minima. Therefore

XXIV. *Under hypothesis* C, *there exists a* t_1 *such that for* $t > t_1$, *the velocity* v *decreases steadily toward zero as* t *increases.*[*]

If v has an infinity of extremes, the points at which they occur must consequently have a limiting point t_1 at finite distance. By Rolle's theorem, the zeros of v'' separate those of v', and consequently t_1 is a limiting point of zeros of v'', and similarly of as many of the higher derivatives as may exist. By continuity, we therefore have $v'(t_1) = 0, v''(t_1) = 0, v'''(t_1) = 0, \cdots$.

Assuming the second derivatives of $G(v)$ and $H(y)$ to exist, v''' exists. Differentiating (37) twice, substituting the values of x''' and y''' obtained from (1), making $v' = v'' = 0$, and eliminating E by $v' = 0$ or (49), it is shown without difficulty that

$$\tfrac{1}{2}(v^2)''' = -2h(y)g^2\frac{y'^3}{v^2} + h(y)gy' + h(y)^2gy'^3 - g^3\frac{y'^3}{v^4} + g^3\frac{y'}{v^2} - h'(y)g\frac{y'^3}{v^2}.$$

Equating this expression to zero, and making $v'' = 0$ in (50), we obtain after some algebraic reductions

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[*] Under the assumption (contrary to C) that $h(y) \leqq 0$, it follows from (50) that $v'' > 0$, that is, every extreme is a minimum. The velocity has therefore one minimum (since it decreases on the rising branch) and no maximum, or else it decreases steadily from $t = 0$ onward.

$$2h(y)^2 - \frac{h'(y)}{v^2} = 0.$$

When $h'(y) \leqq 0$, this equation is impossible, so that XXIV may be replaced by the following:

XXV. *When, in addition to hypothesis* C, *we have* $h'(y) \leqq 0$, *the velocity* v *has only a finite number of extremes.*

In particular, this is true under the assumption made in practice, $h(y) = h =$ a positive constant.

Since v decreases on the rising branch, the first extreme is necessarily a minimum, and from $v \to 0$ as $t \to \infty$ it follows that the last extreme must be a maximum. In many computed trajectories, a minimum followed by a maximum occurs for $y > 0$, or before the projectile strikes the ground (the part of the falling branch where $y < 0$ being of course of no practical interest). In no case, however, has a second minimum been discovered, and it is therefore plausible enough that v has only one minimum and one maximum, even for $y < 0$. The question of proving or disproving this conjecture remains open.

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